

The isoperimetric problem for Hölderian curves

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Abstract

We prove a necessary stationary condition for non-differentiable isoperimetric variational problems with scale derivatives, defined on the class of Hölder continuous functions.

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1 Introduction

An analogue of differentiable calculus for Hölder continuous functions has been recently developed by J. Cresson, by substituting the classical notion of derivative by a new complex operator, called the *scale derivative* [2]. A Leibniz rule similar to the classical one is proved, and with it a generalized Euler-Lagrange equation, valid for nonsmooth curves, is obtained [2]. The new calculus of variations find applications in scale-relativity theory, and some applications are given to Hamilton's principle of least action and to nonlinear Schrödinger equations [1, 2, 3].

In this note we introduce the isoperimetric problem for Hölder continuous curves in Cresson's setting. Section 2 reviews the quantum calculus of J. Cresson, fixing some typos found in [2]. Main results are given in Section 3, where the non differentiable isoperimetric problem is formulated and respective stationary condition proved (see Theorem 4). We end with Section 4, illustrating the applicability of our Theorem 4 to a simple example that has an Hölder continuous extremal, which is not differentiable in the classical sense.

2 Preliminaries

In this section we review the quantum calculus [1, 2], which extends the classical differential calculus to nonsmooth continuous curves. As usual, we denote by C^0 the set of continuous real valued functions defined on \mathbb{R} .

Definition 1. ([2]) Let $f \in C^0$ and $\epsilon > 0$. The ϵ -left and ϵ -right quantum derivatives are defined by

$$\Delta_\epsilon^- f(x) = -\frac{f(x-\epsilon) - f(x)}{\epsilon} \quad \text{and} \quad \Delta_\epsilon^+ f(x) = \frac{f(x+\epsilon) - f(x)}{\epsilon},$$

respectively. In short, we write $\Delta_\epsilon^\sigma f(x)$, $\sigma = \pm$.

Next concept generalizes the derivative for continuous functions, not necessarily smooth.

Definition 2. (cf. [2]) Let $f \in C^0$ and $\epsilon > 0$. The ϵ scale derivative of f at x is defined by

$$\frac{\square_\epsilon f}{\square x}(x) = \frac{1}{2}(\Delta_\epsilon^+ f(x) + \Delta_\epsilon^- f(x)) - i\frac{1}{2}(\Delta_\epsilon^+ f(x) - \Delta_\epsilon^- f(x)), \quad i^2 = -1. \quad (1)$$

If f is a C^1 function, and if we take the limit as $\epsilon \rightarrow 0$ in (1), we obtain $f'(x)$. To simplify, when there is no danger of confusion, we will write $\square_\epsilon f$ instead of $\square_\epsilon f / \square x$. For complex valued functions, we define

$$\frac{\square_\epsilon f}{\square x}(x) = \frac{\square_\epsilon \text{Re}(f)}{\square x}(x) + i \frac{\square_\epsilon \text{Im}(f)}{\square x}(x).$$

We now collect the results needed to this work. First the Leibniz rule for quantum calculus:

Theorem 1. (cf. [2]) Given $f, g \in C^0$ and $\epsilon > 0$, one has

$$\square_\epsilon(f \cdot g) = \square_\epsilon f \cdot g + f \cdot \square_\epsilon g + i\frac{\epsilon}{2}(\square_\epsilon f \square_\epsilon g - \square_\epsilon f \square_\epsilon g - \square_\epsilon f \square_\epsilon g - \square_\epsilon f \square_\epsilon g), \quad (2)$$

where $\square_\epsilon f$ is the complex conjugate of $\square_\epsilon f$.

If f and g are both differentiable, we obtain the Leibniz rule $(f \cdot g)' = f' \cdot g + f \cdot g'$ from (2), taking the limit as $\epsilon \rightarrow 0$.

Definition 3. Let $f \in C^0$, and $\alpha \in (0, 1)$ be a real number. We say that f is Hölderian of Hölder exponent α if there exists a constant c such that, for all $\epsilon > 0$, and all $x, x' \in \mathbb{R}$ such that $|x - x'| \leq \epsilon$,

$$|f(x) - f(x')| \leq c\epsilon^\alpha.$$

We denote by H^α the set of Hölderian functions with Hölder exponent α .

From now on, we assume that $\alpha \in (0, 1)$ is fixed, and ϵ is a sufficiently small parameter, $0 < \epsilon \ll 1$. Let

$$C_\epsilon^\alpha(a, b) = \{y : [a - \epsilon, b + \epsilon] \rightarrow \mathbb{R} \mid y \in H^\alpha\}.$$

A functional is a function $\Phi : C_\epsilon^\alpha(a, b) \rightarrow \mathbb{C}$. We study the class of functionals Φ of the form

$$\Phi(y) = \int_a^b f(x, y(x), \square_\epsilon y(x)) dx, \quad (3)$$

where $f : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{C}$ is a C^1 function, called the Lagrangian. We assume that the Lagrangian satisfies

$$\|Df(x, y(x), \square_\epsilon y(x))\| \leq C,$$

where C is a positive constant, D denotes the differential, and $\|\cdot\|$ is a norm for matrices.

If we consider the class of differentiable functions $y \in C^1$, we obtain the classical functional

$$\Phi(y) = \int_a^b f(x, y(x), \dot{y}(x)) dx$$

of the calculus of variations when ϵ goes to zero.

The methods to solve problems of the calculus of variations admit a common variational approach: we consider a class of functions $\eta(x)$ such that $\eta(a) = 0 = \eta(b)$; and admissible functions $\bar{y} = y + \epsilon_1 \eta$ on the neighborhood of y . For ϵ_1 sufficiently small, \bar{y} is infinitely near y and satisfies given boundary conditions $\bar{y}(a) = y(a)$ and $\bar{y}(b) = y(b)$. For our purposes, we need another assumption about functions η .

Definition 4. ([2]) Let $y \in C_\epsilon^\alpha(a, b)$. A variation \bar{y} of y is a curve of the form $\bar{y} = y + h$, where $h \in C_\epsilon^\beta(a, b)$, $\beta \geq \alpha 1_{[1/2, 1]} + (1 - \alpha) 1_{]0, 1/2[}$, and $h(a) = 0 = h(b)$.

The minimal condition on β is to ensure that the variation curve \bar{y} is still on $C_\epsilon^\alpha(a, b)$.

Definition 5. ([2]) A functional Φ is called differentiable on $C_\epsilon^\alpha(a, b)$ if for all variations $\bar{y} = y + h$, $h \in C_\epsilon^\beta(a, b)$,

$$\Phi(y + h) - \Phi(y) = F_y(h) + R_y(h),$$

where F_y is a linear operator and $R_y(h) = O(h^2)$.

Theorem 2. (*cf.* [2]) For all $\epsilon > 0$, the functional Φ defined by (3) is differentiable, and its derivative is

$$F_y(h) = \int_a^b \left[\frac{\partial f}{\partial y}(x, y(x), \square_\epsilon y(x)) - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y}(x, y(x), \square_\epsilon y(x)) \right) \right] h(x) dx \\ + \int_a^b \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y} h(x) \right) dx + iR_y(h)$$

with

$$R_y(h) = -\frac{\epsilon}{2} \int_a^b [\square_\epsilon f_\epsilon(x) \square_\epsilon h(x) - \boxminus_\epsilon f_\epsilon(x) \square_\epsilon h(x) - \square_\epsilon f_\epsilon(x) \boxminus_\epsilon h(x) \\ - \boxminus_\epsilon f_\epsilon(x) \boxminus_\epsilon h(x)] dx$$

where

$$f_\epsilon(x) = \frac{\partial f}{\partial \square_\epsilon y}(x, y(x), \square_\epsilon y(x)).$$

Definition 6. ([2]) Let $a_p(\epsilon)$ be a real or complex valued function, with parameter p . We denote by $[\cdot]_\epsilon$ the (unique) linear operator defined by

$$a_p(\epsilon) - [a_p(\epsilon)]_\epsilon \rightarrow_{\epsilon \rightarrow 0} 0 \quad \text{and} \quad [a_p(\epsilon)]_\epsilon = 0 \quad \text{if} \quad \lim_{\epsilon \rightarrow 0} a_p(\epsilon) = 0.$$

Definition 7. ([2]) We say that y is an extremal curve for the functional (3) on $C_\epsilon^\beta(a, b)$, if $[F_y(h)]_\epsilon = 0$ for all $\epsilon > 0$ and $h \in C_\epsilon^\beta(a, b)$.

The main result of [2] is a version of the Euler-Lagrange equation for nonsmooth curves:

Theorem 3. ([2]) The curve y is an extremal for the functional (3) on $C_\epsilon^\beta(a, b)$ if and only if

$$\left[\frac{\partial f}{\partial y}(x, y(x), \square_\epsilon y(x)) - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y}(x, y(x), \square_\epsilon y(x)) \right) \right]_\epsilon = 0$$

for every $\epsilon > 0$.

3 Main results

The isoperimetric problem is one of the most ancient optimization problems. One seeks to find a continuously differentiable curve $y = y(x)$, satisfying

given boundary condition $y(a) = a_0$ and $y(b) = b_0$, which minimizes or maximizes a given functional

$$I(y) = \int_a^b f(x, y(x), \dot{y}(x)) dx,$$

for which a second given functional

$$G(y) = \int_a^b g(x, y(x), \dot{y}(x)) dx$$

possesses a given prescribed value K . The classical method to solve this problem involves a Lagrange multiplier λ and consider the problem of extremizing the functional

$$\int_a^b (f - \lambda g) dx$$

using the respective Euler-Lagrange equation. In scale calculus we have an additional problem, because functionals I and G take complex values and so the Lagrange multiplier method must be adapted. We will assume that $\|Dg(\cdot)\|$ is finite.

For our main theorem, we need the following lemma.

Lemma 1. *If $\lim_{\epsilon \rightarrow 0}(a_p(\epsilon))$ and $\lim_{\epsilon \rightarrow 0}(b_p(\epsilon))$ are both finite, then*

$$[a_p(\epsilon) \cdot b_p(\epsilon)]_\epsilon = [a_p(\epsilon)]_\epsilon \cdot [b_p(\epsilon)]_\epsilon.$$

Proof. Since $\lim_{\epsilon \rightarrow 0}(a_p(\epsilon))$ and $\lim_{\epsilon \rightarrow 0}(b_p(\epsilon))$ are finite, then $\lim_{\epsilon \rightarrow 0}[a_p(\epsilon)]_\epsilon$ and $\lim_{\epsilon \rightarrow 0}[b_p(\epsilon)]_\epsilon$ are also finite. Moreover,

1.

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0}(a_p(\epsilon) \cdot b_p(\epsilon) - [a_p(\epsilon)]_\epsilon \cdot [b_p(\epsilon)]_\epsilon) \\ &= \lim_{\epsilon \rightarrow 0}((a_p(\epsilon) - [a_p(\epsilon)]_\epsilon) \cdot b_p(\epsilon) + [a_p(\epsilon)]_\epsilon \cdot (b_p(\epsilon) - [b_p(\epsilon)]_\epsilon)) = 0. \end{aligned}$$

2. If $\lim_{\epsilon \rightarrow 0}(a_p(\epsilon) \cdot b_p(\epsilon)) = 0$, then $\lim_{\epsilon \rightarrow 0}(a_p(\epsilon)) = 0$ or $\lim_{\epsilon \rightarrow 0}(b_p(\epsilon)) = 0$. Therefore, $[a_p(\epsilon)]_\epsilon = 0$ or $[b_p(\epsilon)]_\epsilon = 0$ and so $[a_p(\epsilon)]_\epsilon \cdot [b_p(\epsilon)]_\epsilon = 0$.

□

Definition 8. *Given a constraint functional $G(y) = K$ and a curve \bar{y} , we say that \bar{y} is an extremal curve for the functional $I(y) = \int_a^b f(x, y(x), \square_\epsilon y(x)) dx$*

subject to the constraint $G(y) = K$, if whenever $\hat{y} = \bar{y} + \sum_k h_k$, $h_k \in C_\epsilon^\beta(a, b)$, is a variation satisfying the constraint $G(\hat{y}) = K$, then

$$[F_{\bar{y}}(h_k)]_\epsilon = \int_a^b \left[\frac{\partial f}{\partial y}(x, \bar{y}(x), \square_\epsilon \bar{y}(x)) - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y}(x, \bar{y}(x), \square_\epsilon \bar{y}(x)) \right) \right]_\epsilon h_k(x) dx = 0$$

for all $\epsilon > 0$ and for all k .

Theorem 4. Let $\bar{y} \in C_\epsilon^\alpha(a, b)$. Suppose that \bar{y} is an extremal for the functional

$$\begin{aligned} I : C_\epsilon^\alpha(a, b) &\rightarrow \mathbb{C} \\ y &\mapsto \int_a^b f(x, y(x), \square_\epsilon y(x)) dx \end{aligned}$$

on $C_\epsilon^\beta(a, b)$, subject to the boundary conditions $y(a) = a_0$, $y(b) = b_0$ and the integral constraint

$$G(y) = \int_a^b g(x, y(x), \square_\epsilon y(x)) dx = K,$$

where $K \in \mathbb{C}$ is a given constant. If

1. \bar{y} is not an extremal for G ;

2. and $\lim_{\epsilon \rightarrow 0} \max_{x \in [a, b]} \left| \left(\frac{\partial f}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y} \right) \right) \right|_{(x, \bar{y}(x), \square_\epsilon \bar{y}(x))} \Big|$ and $\lim_{\epsilon \rightarrow 0} \max_{x \in [a, b]} \left| \left(\frac{\partial g}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial g}{\partial \square_\epsilon y} \right) \right) \right|_{(x, \bar{y}(x), \square_\epsilon \bar{y}(x))} \Big|$ are both finite;

then there exists $\lambda \in \mathbb{R}$ such that

$$\left[\left(\frac{\partial L}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial L}{\partial \square_\epsilon y} \right) \right) \right]_{(x, \bar{y}(x), \square_\epsilon \bar{y}(x))} \Big|_\epsilon = 0,$$

where $L = f - \lambda g$. In other words, \bar{y} is an extremal for L .

Remark 1. Hypothesis 2 of Theorem 4 is trivially satisfied in the case where the admissible curves are smooth.

Proof. To short, let $u = (x, \bar{y}(x), \square_\epsilon \bar{y}(x))$. Consider the two-parameter family of variations

$$\hat{y} = \bar{y} + \epsilon_1 \eta_1 + \epsilon_2 \eta_2,$$

such that $\eta_1, \eta_2 \in C_\epsilon^\beta(a, b)$, $\beta \geq \alpha 1_{[1/2, 1]} + (1 - \alpha) 1_{]0, 1/2[}$, $\eta_1(a) = 0 = \eta_1(b)$, $\eta_2(a) = 0 = \eta_2(b)$, and $\epsilon_1, \epsilon_2 \in B_r(0)$, with r sufficiently small. Then, $\hat{y}(a) = a_0$ and $\hat{y}(b) = b_0$, as prescribed, for all values of the parameters ϵ_1 and ϵ_2 . It is easy to see that $\hat{y} \in C_\epsilon^\alpha(a, b)$.

1. If we fix two curves η_1 and η_2 , we can consider the functions \bar{I} and \bar{G} with two variables ϵ_1 and ϵ_2 , defined by

$$\bar{I}(\epsilon_1, \epsilon_2) = \int_a^b f(x, \bar{y}(x) + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, \square_\epsilon \bar{y}(x) + \epsilon_1 \square_\epsilon \eta_1 + \epsilon_2 \square_\epsilon \eta_2) dx$$

and

$$\bar{G}(\epsilon_1, \epsilon_2) = \int_a^b g(x, \bar{y}(x) + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, \square_\epsilon \bar{y}(x) + \epsilon_1 \square_\epsilon \eta_1 + \epsilon_2 \square_\epsilon \eta_2) dx.$$

Let $\bar{\bar{G}} = \bar{G} - K$.

2. We have $\nabla \bar{\bar{G}}(0, 0) \neq 0$. Indeed, since g is a smooth function, $\bar{\bar{G}}$ is also smooth and

$$\begin{aligned} \left. \frac{\partial \bar{\bar{G}}}{\partial \epsilon_1} \right|_{(0,0)} &= \int_a^b \left(\eta_1 \left. \frac{\partial g}{\partial y} \right|_u + \square_\epsilon \eta_1 \left. \frac{\partial g}{\partial \square_\epsilon y} \right|_u \right) dx \\ &= \int_a^b \left(\left. \frac{\partial g}{\partial y} \right|_u - \frac{\square_\epsilon}{\square x} \left(\left. \frac{\partial g}{\partial \square_\epsilon y} \right|_u \right) \right) \eta_1 dx \\ &\quad + \int_a^b \frac{\square_\epsilon}{\square x} \left(\left. \frac{\partial g}{\partial \square_\epsilon y} \right|_u \cdot \eta_1 \right) dx \\ &\quad - i \frac{\epsilon}{2} \int_a^b [\square_\epsilon g_\epsilon \square_\epsilon \eta_1 - \boxminus_\epsilon g_\epsilon \square_\epsilon \eta_1 - \square_\epsilon g_\epsilon \boxminus_\epsilon \eta_1 - \boxminus_\epsilon g_\epsilon \boxminus_\epsilon \eta_1] dx, \end{aligned}$$

where

$$g_\epsilon = \left. \frac{\partial g}{\partial \square_\epsilon y} \right|_u.$$

Since

$$\lim_{\epsilon \rightarrow 0} \int_a^b \frac{\square_\epsilon}{\square x} \left(\left. \frac{\partial g}{\partial \square_\epsilon y} \right|_u \cdot \eta_1(x) \right) dx = 0$$

and

$$\lim_{\epsilon \rightarrow 0} \epsilon \int_a^b (Op_\epsilon g_\epsilon Op'_\epsilon \eta_1) dx = 0,$$

where Op_ϵ and Op'_ϵ is equal to \square_ϵ and \boxminus_ϵ (**cf.** [2, Lemma 3.2]), it follows that

$$\left[\left. \frac{\partial \bar{\bar{G}}}{\partial \epsilon_1} \right|_{(0,0)} \right]_\epsilon = \int_a^b \left[\left. \frac{\partial g}{\partial y} \right|_u - \frac{\square_\epsilon}{\square x} \left(\left. \frac{\partial g}{\partial \square_\epsilon y} \right|_u \right) \right]_\epsilon \eta_1(x) dx.$$

Since \bar{y} is not an extremal of G , there exists a curve η_1 such that

$$\left[\frac{\partial \bar{G}}{\partial \epsilon_1} \Big|_{(0,0)} \right]_{\epsilon} \neq 0.$$

Therefore, by the definition of $[\cdot]_{\epsilon}$, we conclude that

$$\frac{\partial \bar{G}}{\partial \epsilon_1} \Big|_{(0,0)} \neq 0.$$

3. We can choose $\epsilon_2 \eta_2$ in order to satisfy the isoperimetric condition. Since $\nabla \bar{G}(0,0) \neq 0$ and $\bar{G}(0,0) = 0$, by the implicit function theorem, there exists a function $\epsilon_1 := \epsilon_1(\epsilon_2)$ defined on a neighbourhood of zero such that

$$\bar{G}(\epsilon_1(\epsilon_2), \epsilon_2) = 0.$$

4. We now adapt the Lagrange multiplier method. Since $\bar{G}(\epsilon_1(\epsilon_2), \epsilon_2) = 0$, for any ϵ_2 , then

$$0 = \frac{d}{d\epsilon_2} \bar{G}(\epsilon_1(\epsilon_2), \epsilon_2) = \frac{d\epsilon_1}{d\epsilon_2} \cdot \frac{\partial \bar{G}}{\partial \epsilon_1} + \frac{\partial \bar{G}}{\partial \epsilon_2}$$

and so, as ϵ goes to zero,

$$\frac{d\epsilon_1}{d\epsilon_2} \Big|_0 = - \frac{\int_a^b \left(\frac{\partial g}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial g}{\partial \square_{\epsilon} y} \Big|_u \right) \right) \eta_2(x) dx + \int_a^b \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial g}{\partial \square_{\epsilon} y} \Big|_u \cdot \eta_2 \right) dx - [\dots]}{\int_a^b \left(\frac{\partial g}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial g}{\partial \square_{\epsilon} y} \Big|_u \right) \right) \eta_1(x) dx + \int_a^b \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial g}{\partial \square_{\epsilon} y} \Big|_u \cdot \eta_1 \right) dx - [\dots]}$$

is finite. Observe that

$$\lim_{\epsilon \rightarrow 0} \frac{\partial \bar{I}}{\partial \epsilon_1} \Big|_u = \lim_{\epsilon \rightarrow 0} \int_a^b \left(\frac{\partial f}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial f}{\partial \square_{\epsilon} y} \Big|_u \right) \right) \eta_1(x) dx$$

and

$$\lim_{\epsilon \rightarrow 0} \frac{\partial \bar{G}}{\partial \epsilon_1} \Big|_u = \lim_{\epsilon \rightarrow 0} \int_a^b \left(\frac{\partial g}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial g}{\partial \square_{\epsilon} y} \Big|_u \right) \right) \eta_1(x) dx$$

are also finite. Let us prove that

$$\frac{d}{d\epsilon_2} [\bar{I}(\epsilon_1(\epsilon_2), \epsilon_2)]_0 \Big|_{\epsilon} = 0. \quad (4)$$

A direct calculation shows that

$$\begin{aligned}
\frac{d}{d\epsilon_2} [\bar{I}(\epsilon_1(\epsilon_2), \epsilon_2)|_0]_\epsilon &= \left[\frac{d\epsilon_1}{d\epsilon_2} \frac{\partial \bar{I}}{\partial \epsilon_1} + \frac{\partial \bar{I}}{\partial \epsilon_2} \right]_\epsilon \\
&= \left[\frac{d\epsilon_1}{d\epsilon_2} \right]_\epsilon \left[\frac{\partial \bar{I}}{\partial \epsilon_1} \right]_\epsilon + \left[\frac{\partial \bar{I}}{\partial \epsilon_2} \right]_\epsilon \\
&= \left[\frac{d\epsilon_1}{d\epsilon_2} \right]_\epsilon \int_a^b \left[\frac{\partial f}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y} \right) \right]_\epsilon \eta_1 dx \\
&\quad + \int_a^b \left[\frac{\partial f}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y} \right) \right]_\epsilon \eta_2 dx \\
&= \left[\frac{d\epsilon_1}{d\epsilon_2} \right]_\epsilon [F_{\bar{y}}(\eta_1)]_\epsilon + [F_{\bar{y}}(\eta_2)]_\epsilon = 0
\end{aligned}$$

since \bar{y} is an extremal of I subject to the constraint $G = K$. On the other hand, for any ϵ_2 , we also have $\left[\bar{G}(\epsilon_1(\epsilon_2), \epsilon_2) \right]_\epsilon = 0$. Therefore,

$$0 = \frac{d}{d\epsilon_2} [\bar{G}(\epsilon_1(\epsilon_2), \epsilon_2)]_\epsilon = \left[\frac{d\epsilon_1}{d\epsilon_2} \right]_\epsilon \cdot \left[\frac{\partial \bar{G}}{\partial \epsilon_1} \right]_\epsilon + \left[\frac{\partial \bar{G}}{\partial \epsilon_2} \right]_\epsilon$$

and so

$$\left[\frac{d\epsilon_1}{d\epsilon_2} \right]_\epsilon = - \frac{\left[\frac{\partial \bar{G}}{\partial \epsilon_2} \right]_\epsilon}{\left[\frac{\partial \bar{G}}{\partial \epsilon_1} \right]_\epsilon}.$$

Using condition (4), we have

$$\left| \begin{array}{c} \left[\frac{\partial \bar{I}}{\partial \epsilon_1} \right]_\epsilon \\ \left[\frac{\partial \bar{I}}{\partial \epsilon_2} \right]_\epsilon \end{array} \right|_\epsilon \begin{array}{c} \left[\frac{\partial \bar{G}}{\partial \epsilon_1} \right]_\epsilon \\ \left[\frac{\partial \bar{G}}{\partial \epsilon_2} \right]_\epsilon \end{array} \right|_\epsilon = 0.$$

Since $\left[\frac{\partial \bar{G}}{\partial \epsilon_1} \right]_\epsilon \neq 0$, we conclude that there exists some real λ such that

$$\left(\left[\frac{\partial \bar{I}}{\partial \epsilon_1} \right]_\epsilon, \left[\frac{\partial \bar{I}}{\partial \epsilon_2} \right]_\epsilon \right) = \lambda \left(\left[\frac{\partial \bar{G}}{\partial \epsilon_1} \right]_\epsilon, \left[\frac{\partial \bar{G}}{\partial \epsilon_2} \right]_\epsilon \right).$$

5. In conclusion, since

$$\begin{aligned}
0 &= \left[\frac{\partial}{\partial \epsilon_2} (\bar{I} - \lambda \bar{G}) \Big|_{(0,0)} \right]_{\epsilon} \\
&= \int_a^b \left[\eta_2 \frac{\partial f}{\partial y} \Big|_u + \square_{\epsilon} \eta_2 \frac{\partial f}{\partial \square_{\epsilon} y} \Big|_u - \lambda \left(\eta_2 \frac{\partial g}{\partial y} \Big|_u + \square_{\epsilon} \eta_2 \frac{\partial g}{\partial \square_{\epsilon} y} \Big|_u \right) \right]_{\epsilon} dx \\
&= \int_a^b \left[\frac{\partial f}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial f}{\partial \square_{\epsilon} y} \Big|_u \right) - \lambda \left(\frac{\partial g}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial g}{\partial \square_{\epsilon} y} \Big|_u \right) \right) \right]_{\epsilon} \eta_2 dx \\
&= \int_a^b \left[\frac{\partial L}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial L}{\partial \square_{\epsilon} y} \Big|_u \right) \right]_{\epsilon} \eta_2 dx
\end{aligned}$$

and η_2 is any curve, we obtain

$$\left[\frac{\partial L}{\partial y} \Big|_u - \frac{\square_{\epsilon}}{\square x} \left(\frac{\partial L}{\partial \square_{\epsilon} y} \Big|_u \right) \right]_{\epsilon} = 0.$$

□

4 An example

Let $f(x, y, v) = (v - \frac{\square_{\epsilon}}{\square x} |x|)^2$. With simple calculations, one proves that

$$\frac{\square_{\epsilon}}{\square x} |x| = \begin{cases} 1 & \text{if } x \geq \epsilon \\ x/\epsilon - i(\epsilon - x)/\epsilon & \text{if } 0 \leq x < \epsilon \\ x/\epsilon - i(\epsilon + x)/\epsilon & \text{if } -\epsilon < x < 0 \\ -1 & \text{if } x \leq -\epsilon \end{cases}$$

Suppose we want to find the extremals for the functional

$$\int_{-1}^1 f(x, y(x), \square_{\epsilon} y(x)) dx \tag{5}$$

subject to the integral constraint

$$\int_{-1}^1 g(x, y(x), \square_{\epsilon} y(x)) dx = \frac{2}{3},$$

where $g(x, y, v) = x + y^2$, and to the boundary conditions $y(-1) = 1 = y(1)$. The (nonsmooth) curve $y = |x|$ satisfies the constraint integral, and the following conditions:

1. $\left[\frac{\partial f}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y} \right) \right]_\epsilon = 0:$
 $\frac{\partial f}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y} \right) = -\frac{\square_\epsilon}{\square x} (2 (\frac{\square_\epsilon}{\square x} |x| - \frac{\square_\epsilon}{\square x} |x|)) = 0.$
2. $\left[\frac{\partial g}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial g}{\partial \square_\epsilon y} \right) \right]_\epsilon \neq 0:$
 $\frac{\partial g}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial g}{\partial \square_\epsilon y} \right) = 2|x|.$
3. $\lim_{\epsilon \rightarrow 0} \max_{x \in [-1,1]} \left| \frac{\partial f}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial f}{\partial \square_\epsilon y} \right) \right| = \lim_{\epsilon \rightarrow 0} 0 = 0.$
4. $\lim_{\epsilon \rightarrow 0} \max_{x \in [-1,1]} \left| \frac{\partial g}{\partial y} - \frac{\square_\epsilon}{\square x} \left(\frac{\partial g}{\partial \square_\epsilon y} \right) \right| = \lim_{\epsilon \rightarrow 0} 2 = 2.$

Observe that, since $y = |x|$ is actually an extremal of (5), we may take $\lambda = 0$.

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